

On the power graph of the direct product of two groups

A. K. Bhuniya and Sajal Kumar Mukherjee

Department of Mathematics, Visva-Bharati, Santiniketan-731235, India.

anjankbhuniya@gmail.com, shyamal.sajalmukherjee@gmail.com

Abstract

The power graph $P(G)$ of a finite group G is the graph with vertex set G and two distinct vertices are adjacent if either of them is a power of the other. Here we show that the power graph $P(G_1 \times G_2)$ of the direct product of two groups G_1 and G_2 is not isomorphic to either of the direct, cartesian and normal product of their power graphs $P(G_1)$ and $P(G_2)$. A new product of graphs, namely generalized product, has been introduced and we prove that the power graph $P(G_1 \times G_2)$ is isomorphic to a generalized product of $P(G_1)$ and $P(G_2)$.

Keywords: finite groups; direct product; power graphs; product of graphs; isomorphism.

AMS Subject Classifications: 05C25

1 Introduction

The directed power graph of a semigroup was defined by Kelarev and Quinn [6]. Then Chakraborty et. al [3] defined the undirected power graph $P(S)$ of a semigroup S as the graph with vertex set S and two distinct vertices a and b are adjacent if either $a^m = b$ or $b^n = a$ for some $m, n \in \mathbb{N}$. There are many articles associating group theoretic behavior of G and the graph theoretic properties of $P(G)$. We refer to the survey [1] for an account of the development on the power graph of groups and semigroups. Success attained by the researchers in this direction influenced others to generalize power graph of groups to strong power graph [8], deleted power graph $P^*(G)$ [7], enhanced power graph [2].

Here we investigate relationship of $P(G_1 \times G_2)$ with $P(G_1)$ and $P(G_2)$. There are many standard products of graphs, already defined, namely, direct product, cartesian product, normal product etc. Here we show that, in general, $P(G_1 \times G_2)$ is not isomorphic to either of these products of $P(G_1)$ and $P(G_2)$. So we introduce a new product of two graphs Γ_1 and Γ_2 , which we call generalized product

of Γ_1 and Γ_2 . Reason behind such naming is that each of the cartesian, direct and normal products is a special case of the generalized product of two graphs.

Our main theorem states that $P(G_1 \times G_2)$ is isomorphic to a generalized product of the power graphs $P(G_1)$ and $P(G_2)$.

2 Main result

We refer to [4] and [9] for the notions on graph theory and to [5] for group theoretic background.

Throughout this article \mathbb{N} denotes the set of natural numbers and $\mathbb{Z}^\# = \mathbb{N} \cup \{0\}$.

Let us recall different standard notions of products of graphs.

Definition 2.1. *Let Γ_1 and Γ_2 be two graphs.*

(i) *The direct product $\Gamma_1 \times \Gamma_2$ of Γ_1 and Γ_2 is defined as follows:*

$$V(\Gamma_1 \times \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2) \text{ and } (g_1, g_2) \sim (g'_1, g'_2) \text{ if and only if } g_1 \sim g'_1 \text{ and } g_2 \sim g'_2.$$

(ii) *The cartesian product $\Gamma_1 \boxtimes \Gamma_2$ of Γ_1 and Γ_2 is defined as follows:*

$$V(\Gamma_1 \boxtimes \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2) \text{ and } (g_1, g_2) \sim (g'_1, g'_2) \text{ if and only if either } g_1 = g'_1 \text{ and } g_2 \sim g'_2 \text{ or } g_1 \sim g'_1 \text{ and } g_2 = g'_2.$$

(iii) *The normal product $\Gamma_1 * \Gamma_2$ of Γ_1 and Γ_2 is defined as follows:*

$$V(\Gamma_1 * \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2) \text{ and } (g_1, g_2) \sim (g'_1, g'_2) \text{ if and only if either } g_1 \sim g'_1 \text{ and } g_2 \sim g'_2 \text{ or } g_1 = g'_1 \text{ and } g_2 \sim g'_2 \text{ or } g_1 \sim g'_1 \text{ and } g_2 = g'_2.$$

Now we show that neither of these notions of product of graphs is enough to catch the relationship of $P(G_1 \times G_2)$ with $P(G_1)$ and $P(G_2)$.

Proposition 2.2. *Let G_1 and G_2 be two nontrivial finite groups. Then $P(G_1 \times G_2)$ is not isomorphic to $P(G_1) \boxtimes P(G_2)$.*

Proof. Suppose, on the contrary that the two graphs, stated in the theorem are isomorphic. Let e_i be the identity element of the group G_i . Let g_1, g_2 be two nonidentity elements of G_1 and G_2 respectively. Then (e_1, e_2) is not adjacent to (g_1, g_2) in $P(G_1) \boxtimes P(G_2)$, whereas $(e_{G_1}, e_{G_2}) \sim (g_1, g_2)$ in $P(G_1 \times G_2)$. A contradiction. \square

Consider $G_1 = G_2 = \mathbb{Z}_2$. Then $P(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has precisely three edges, each edge emanating from the identity of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and connects the remaining three vertices, whereas $P(\mathbb{Z}_2) * P(\mathbb{Z}_2)$ is the complete graph K_4 and $P(\mathbb{Z}_2) \times P(\mathbb{Z}_2)$ is a graph with precisely two edges.

Thus we show that $P(G_1 \times G_2)$ is neither isomorphic to the direct product nor to the normal product of $P(G_1)$ and $P(G_2)$, in general.

So we need to introduce another product of graphs to uncover the relationship of the power graphs $P(G_1)$ and $P(G_2)$ with $P(G_1 \times G_2)$ for any two groups G_1 and G_2 .

Before going into technical details, note the following fact. Let G be a finite group and $a, b \in G$. Suppose that $a \sim b$ in $P(G)$. It is easy to observe that if n is the smallest positive integer such that $a^n = b$, then $\{m \in \mathbb{N} : a^m = b\}$ is the arithmetic progression with initial term n and common difference $o(a)$.

For any two integers a and d , we denote the arithmetic progression with initial term a and common difference d by $AP(a, d)$.

Let Γ be a graph. Then by a generalization on Γ we mean a function $W : A(\Gamma) \cup \Delta \rightarrow \mathbb{Z}^\# \times \mathbb{Z}^\#$, where $A(\Gamma)$ is the arc set of Γ and $\Delta = \{(v, v) : v \text{ is a vertex of } \Gamma\}$.

Definition 2.3. Let (Γ_1, W_1) and (Γ_2, W_2) be two graphs equipped with two generalizations W_1, W_2 respectively. Then the generalized product $\Gamma_1 \times_{W_1 \times W_2} \Gamma_2$ is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and $(g_1, g_2) \sim (g'_1, g'_2)$ if and only if the following two conditions hold simultaneously:

(i) $(g_1, g_2) \neq (g'_1, g'_2)$ and

(ii) $AP(W_1(g_1, g'_1)) \cap AP(W_2(g_2, g'_2)) \cap \mathbb{N} \neq \emptyset$ or $AP(W_1(g'_1, g_1)) \cap AP(W_2(g'_2, g_2)) \cap \mathbb{N} \neq \emptyset$.

If no question of ambiguity arise, then we denote a generalized direct product of two graphs Γ_1 and Γ_2 by $\Gamma_1 \times_W \Gamma_2$.

The following result justifies the name ‘generalized product’.

Theorem 2.4. Each of the direct, cartesian and normal products is a generalized product.

Proof. Let Γ_1 and Γ_2 be two graphs.

(i) Consider the generalizations W_1 of Γ_1 defined by:

$$\begin{aligned} W_1(x, y) &= (1, 1) \text{ if } x \neq y \\ &= (0, 0) \text{ if } x = y. \end{aligned}$$

and W_2 of Γ_2 defined similarly. Then it is easy to verify that $\Gamma_1 \times_W \Gamma_2 = \Gamma_1 \times \Gamma_2$.

(ii) Consider the generalizations W_1 of Γ_1 defined by:

$$\begin{aligned} W_1(x, y) &= (1, 0) \text{ if } x \neq y \\ &= (1, 1) \text{ if } x = y. \end{aligned}$$

and W_2 of Γ_2 defined by:

$$\begin{aligned} W_1(x, y) &= (2, 0) \text{ if } x \neq y \\ &= (1, 1) \text{ if } x = y. \end{aligned}$$

Then $\Gamma_1 \times_W \Gamma_2 = \Gamma_1 \boxtimes \Gamma_2$.

(iii) Consider the generalizations W_1 of Γ_1 defined by:

$$\begin{aligned} W_1(x, y) &= (1, 0) \text{ if } x \neq y \\ &= (1, 1) \text{ if } x = y. \end{aligned}$$

and W_2 of Γ_2 defined similarly. Then $\Gamma_1 \times_W \Gamma_2 = \Gamma_1 * \Gamma_2$.

□

Now, we prove our main theorem.

Theorem 2.5. *For two groups G_1 and G_2 , $P(G_1 \times G_2)$ and $P(G_1) \times_W P(G_2)$ are isomorphic for some choice of generalizations W_1 and W_2 of $P(G_1)$ and $P(G_2)$ respectively.*

Proof. Let us first specify the choice of the generalizations of $P(G_1)$ and $P(G_2)$. We consider the generalization W_1 of $P(G_1)$ defined by:

$$\begin{aligned} W_1(a, b) &= (t, o(a)) \text{ if } t \text{ is the smallest positive integer such that } a^t = b \\ &= (0, 0) \text{ otherwise.} \end{aligned}$$

and the generalization W_2 of $P(G_2)$ defined similarly.

Let $(g_1, g_2) \sim (\acute{g}_1, \acute{g}_2)$ in $P(G_1 \times G_2)$. Then either $(g_1, g_2)^m = (\acute{g}_1, \acute{g}_2)$ or $(\acute{g}_1, \acute{g}_2)^n = (g_1, g_2)$ for some $m, n \in \mathbb{N}$. If $(g_1, g_2)^m = (\acute{g}_1, \acute{g}_2)$ then $g_1^m = \acute{g}_1$ implies that $m \in AP(t, o(g_1))$ where t is the smallest positive integer for which $g_1^t = \acute{g}_1$. Hence $m \in AP(W_1(g_1, \acute{g}_1))$. Similarly $g_2^m = \acute{g}_2$ implies that $m \in AP(W_2(g_2, \acute{g}_2))$. Thus $AP(W_1(g_1, \acute{g}_1)) \cap AP(W_2(g_2, \acute{g}_2)) \cap \mathbb{N} \neq \phi$ and so $(g_1, g_2) \sim (\acute{g}_1, \acute{g}_2)$ in $P(G_1) \times_W P(G_2)$. If $(\acute{g}_1, \acute{g}_2)^n = (g_1, g_2)$ then $AP(W_1(\acute{g}_1, g_1)) \cap AP(W_2(\acute{g}_2, g_2)) \cap \mathbb{N} \neq \phi$ and so $(g_1, g_2) \sim (\acute{g}_1, \acute{g}_2)$ in $P(G_1) \times_W P(G_2)$.

Conversely, let $(g_1, g_2) \sim (\acute{g}_1, \acute{g}_2)$ in $P(G_1) \times_W P(G_2)$. If $m \in AP(W_1(g_1, \acute{g}_1)) \cap AP(W_2(g_2, \acute{g}_2)) \cap \mathbb{N} \neq \phi$, then $(g_1, g_2)^m = (\acute{g}_1, \acute{g}_2)$ and hence $(g_1, g_2) \sim (\acute{g}_1, \acute{g}_2)$ in $P(G_1 \times G_2)$. Similar is the other case. □

Acknowledgement: The second author is partially supported by CSIR-JRF grant.

References

- [1] J. Abawajy, A. V. Kelarev and M. Chowdhury, Power graphs: A survey, *Electron. J. Graph Theory Appl*, **1**(2013), 125-147.
- [2] G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish, F. Shaveisi, On the structure of the power graphs and the enhanced power graphs of a group, *arXiv: 1603.04337v1(2016)[math. CO]*.
- [3] I. Chakrabarty, S. Ghosh and M. K. Sen, Undirected power graphs of semigroups, *Semigroup Forum* , **78**(2009), 410-426.
- [4] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York Inc, 2001.
- [5] T. W. Hungerford, *Algebra*, Graduate Text in Mathematics 73, Springer-Verlag, New York(NY), (1974).
- [6] A. V. Kelarev and S. J. Quinn, A combinatorial property and power graphs of groups, *Contributions to General Algebra*, **12**(Heyn, Klagenfurt, 2000), 229-235.
- [7] A. R. Moghaddanfar, S. Rahbariyan, W. J. Shi, Certain properties of the power graph associated with a finite group, *arXiv: 1310.2032v1(2013)[math. GR]*.
- [8] G. Singh and K. Manilal, Some generalities on power graphs and strong power graphs, *Int. J. Contemp. Math Sciences*, **5**(55)(2010), 2723-2730.
- [9] D. B. West, *Introduction to Graph theory*, 2nd ed., Pearson education, 2001.